A Theory of Household Bargaining in the Presence of Limited Commitment and Public Goods

Zaki Wahhaj*
University of Namur

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Abstract

This paper investigates mutual insurance agreements between two agents in an environment where there is limited commitment and some, but not all, expenditures are public to the agents. This setting is relevant for the study of intra-household insurance, given the significance of public expenditures within the household and the empirical evidence of limited commitment between household members in developing countries. It is shown that in the constrained efficient agreement, private expenditures of the agents do not necessarily co-move, as they do in both the first-best agreement and the constrained efficient agreement when all expenditures are private. In particular, the absence of co-movement of private expenditures within the group do not necessarily imply the absence of mutual insurance; and, indeed, negative correlation in the change in private expenditures over time can be consistent with a cooperative agreement. These results suggest caution in interpreting the correlation of private expenditures of a group of agents as a measure of cooperation and mutual insurance when lack of commitment and public goods are important factors.

*Department of Economics, University of Namur, 8 Rempart de la Vierge, B-5000 Namur, Belgium. Email: zwahhaj@fundp.ac.be.

1 Introduction

It is well recognised in the economics literature that public goods play an important role within the household[1]. Members of a household are bound together by their emotional attachments to one another and the close physical proximity in which they live. Within a family, an important example of public goods are child-related expenditures, which would affect the well-being of both parents. Moreover, members of a household, by definition, live and eat together; such that many of the things they use or consume are inevitably shared among them – a roof, a water faucet, a latrine, and so on – to an extent rarely seen anywhere else in society.

Recent empirical studies provide considerable evidence of inefficiency in intra-household allocation; and this has increasingly brought attention to the possibility of 'limited commitment' within the household[2]. Broadly, the term relates to the idea that if there are limited means to enforce a cooperative agreement among a group of individuals, then this also places limits on what individuals can commit to do in such an agreement. For the most part, the literature on limited commitment has worked with the assumption that all consumption is private. This assumption may be reasonably accurate in the context of informal insurance between separate households in a village. However, it becomes highly questionable when the object is to understand behaviour *within* households. Given the importance of public goods in the latter case, it is important to answer the question whether the presence or absence of public goods makes any important qualitative differences in a model of limited commitment, a question that has not been considered in the literature thus far.

This paper demonstrates that the answer is yes for a reason that is simple and intuitive. Co-movement in consumption is a fundamental feature of full insurance, and of partial insurance under limited commitment when all consumption is private[3]. However, we show that if public goods are present in a household characterised by limited commitment, co-movement in the (private) consumption of household members is not a necessary feature of intra-household insurance. In particular, some household members may experience an *increase* in private consumption when the household suffers an adverse income shock (henceforth, we refer to this result as 'perverse' insurance).

To understand the intuition behind this result, consider the following story. In a rural household, a husband and a wife farm on separate plots of land. They face idiosyncratic risks, so that there is scope for mutual insurance between them. However, because of lack of commitment, they will not insure each other fully. If all consumption within the household is private, we would observe co-movement in the couple's

[1]See, for example, a survey in Bergstrom (1995) and Deaton (1997).

[2]For example, Duflo and Udry (2003), Goldstein (2002) and Dercon and Krishnan (2000) find, for different parts of Africa, that men and women living in the same household do not fully share risk. Duflo and Udry (2003) suggest that their results may be explained by limited commitment. Similarly, Dubois and Ligon (2003) find, for farm households in the Philippines, that food consumption patterns are inconsistent with full risk-sharing within the household. Udry (1996) finds evidence of inefficiency in the allocation of productive resources across farm plots owned by the same household.

[3]Diamond (1967) and Wilson (1968) derive the result for the full insurance model. Kocherlakota (1996) obtains the result for efficient risk-sharing in an environment with limited commitment and no public goods.

consumption but a shock to one's own income would have a larger effect on one's own consumption than a shock to the spouse's income. This is the result obtained from the standard model of limited commitment (where all consumption is private).

Now suppose that the couple has children, and any child-related expenditures provide utility to both parents. One year, when the wife has had a particularly bad shock, and the husband is unwilling to provide her full insurance, she cannot feed the children as well as usual. The husband is upset at this for he too cares about the well-being of the children. So as to compensate him, the wife accepts that he would not have to spend money to pave the path to the homestead as they had originally agreed. Thus, he would have more to spend at the local bar this year although the wife has just had a bad shock. Co-movement in consumption has broken down, although the couple is still engaged in a cooperative agreement. It is possible to compensate the husband in this manner because, in a pareto efficient allocation, it must be that his marginal utility from expenditure on private goods is always higher than that on public goods.

This result implies that the absence of co-movement in private consumption shoud not be interpreted as the absence of intra-household insurance, and furthermore that a *negative* correlation in the private consumption of household members can still be consistent with a cooperative agreement. Some evidence of such negative correlation exists in the empirical literature. Goldstein (2002) finds, for agricultural households in southern Ghana, that the private consumption of husbands and wives are negatively correlated. Dercon and Krishnan (2000) finds for agricultural households in Ethiopia that the nutritional status of at least some household members are positively affected by an adverse shock to household income.

In neither of these papers do the authors provide any possible explanation for their counter-intuitive results. Goldstein interprets the absence of co-movement in the consumption of household members as evidence of the absence of intra-household insurance[4]. The main result in this paper indicate that such an interpretation is potentially erroneous.

More generally, the result has the following implication for future work on intra-household insurance. The literature on informal insurance has often used co-movement in the consumption of a group of individuals as a test of the presence and extent of insurance within the group[5]. This practice should not be applied wholesale to measure the extent of insurance within a household since, as this paper argues, co-movement in private consumption is not a necessary feature of informal insurance in an environment where public goods and lack of commitment are important factors.

The idea of limited commitment has been explored and developed extensively in the literature. It appears

[4]To test whether household public goods serve as a channel for intra-household insurance, Goldstein also estimates the effect of individual shocks on child-related expenditures. The effect of agricultural shocks on these expenditures are close to zero, which provides some further evidence of the absence of intra-household insurance; however, the effect of illness shocks are imprecisely estimated, such that it is not possible to rule out that adults lower child-related expenditures when they suffer from adverse health shocks.

[5]The seminal work in this literature is Townsend (1994), which rejects the full insurance model for households in rural India, but finds that household consumption co-moves with village average consumption.

in Kimball (1988) as a possible basis of mutual insurance schemes in a rural setting; it is also central to Thomas and Worrall (1988), which examines long-term wage contracts between a risk-neutral firm and a risk-averse worker where, at any point in time, either party can renege and contract at a spot market wage. Coate and Ravallion (1993) characterise the conditions under which the first-best insurance agreement is self-enforcing, for a setting with two risk-averse agents. For the same setting, Kocherlakota (1996) characterise constrained efficient agreements, and examine the long-run dynamics of such agreements; and Gauthier and Poitevin (1997) show that if agents have the ability to make ex-ante payments, this can lead to improved efficiency for self-enforcing agreements. Ligon, Thomas and Worrall (2002) show that the constrained efficient agreements are characterised by a simple updating rule; specifically, that for each state of nature, there is a time-invariant interval for the ratio of marginal utilities; and in each period, the ratio of marginal utilities adjusts by the smallest amount necessary to bring it into the current interval. Ligon (2002) develops an axiomatic approach to household bargaining where the Nash bargaining solution is modified to capture the idea of limited commitment. Unlike the present paper, this literature has focused on a setting where all consumption is private.

2 A Model of Limited Commitment

Consider a household consisting of two individuals, A and B. In each period, the household may devote resources to three different types of goods: x^A, x^B, and z ; x^A is composed of private goods consumed by A and x^B those consumed by B. Both individuals derive utility from expenditure on good z ; i.e. the good is public to the household. The per-period utilities of the two individuals are given by $u^A\left(x^A, z\right)$ and $u^B(x^B, z)$ respectively where u^i is increasing, and strictly concave in x^i and z for $i \in \{A, B\}$. The individuals each receive stochastic income streams $\left\{y^A\left(t\right)\right\}_{t=0}^{\infty}$ and $\left\{y^B\left(t\right)\right\}_{t=0}^{\infty}$ which depend on the current state of nature, s, drawn from the set $\{1, 2, ..N\}$. The distribution of s is i.i.d. with $Pr[s(t) = i] = \pi_i$. We assume there is no scope for saving.

We also assume that $\frac{\partial^2 u^i}{\partial x^i \partial z} \geq 0$, for $i \in \{A, B\}$. This ensures, in particular, that increased spending on public goods within the household would not tempt an individual to convert some of her private expenditures into public expenditures, a possibility which cannot be ruled out when the cross-partial between the two goods is negative. The assumption will enable us to apply methods of monotone comparative statics when considering the effect of income shocks on the allocation of the resources within the household.

We consider the setting where the household is characterised by lack of commitment; i.e. there is no external mechanism to enforce an agreement, and either individual can renege in any period if she finds it in her interest to do so. If an individual does renege on her agreement, then she receives her outside option. For the present analysis, the nature of this outside option is not important. Two realistic possibilities in the context of intra-household bargaining are divorce and non-cooperative behaviour within the marriage. The value of the outside option to each person k in a period where the income levels to A and B are y^A and y^B respectively will be written as

$$d^k\left(y^A, y^B\right) + \beta v^k_{aut}$$

where $d^k\left(y^A, y^B\right)$ is the utility derived from the outside option in the current period, a value which ought to depend on current income levels; v^k_{aut} is the future expected utility from pursuing the outside option and does not depend on current income levels as we have assumed that the income process is independent of history; and β is the discount rate.

We shall impose some structure on the function $d^k\left(y^A, y^B\right)$. For this purpose, we require the following definitions. Let $\mu^k(y^A, y^B, \lambda)$, where $\lambda \in (0, \infty)$, be the current-period utility that person k receives from the allocation given by the solution to the following maximisation problem:

$$\begin{aligned} \max_{(x^A, x^B, z)} &: \lambda u^A\left(x^A, z\right) + u^B\left(x^B, z\right) \\ \text{s.t.} &: x^A + x^B + z \leq y^A + y^B \end{aligned} \tag{1}$$

Let $\lambda^k\left(y^A, y^B\right)$ be the ratio of Pareto weights that yields, in the maximisation problem in (1), a current-period utility for person k equal to that obtained from his outside option; i.e. $\lambda = \lambda^k\left(y^A, y^B\right)$ solves the following equation for $k \in \{A, B\}$:

$$\mu^k\left(y^A, y^B, \lambda\right) = d^k\left(y^A, y^B\right)$$

We make the following assumptions about the functions $d^A(.)$ and $d^B(.)$.

Assumption 1 *For each $s \in S$, $\exists \lambda$ such that $\mu^k\left(y^A_s, y^B_s, \lambda\right) > d^k\left(y^A_s, y^B_s\right)$, $k \in \{A, B\}$*

Assumption 2 *$\mu^A(y^A, y^B, \lambda) - d^A(y^A, y^B)$ is decreasing in y^A and increasing in y^B for $\lambda \leq \lambda^A\left(y^A, y^B\right)$. $\mu^B(y^A, y^B, \lambda) - d^B(y^A, y^B)$ is decreasing in y^B and increasing in y^A for $\lambda \geq \lambda^B\left(y^A, y^B\right)$.*

Assumption 3 *$\frac{\partial d^k}{\partial y^k}\left(y^A, y^B\right) \leq \frac{\partial u^k}{\partial x^k}\left(x^k, z\right)$ where $\left(x^k, z\right)$ is obtained from an efficient allocation in which $u^k\left(x^k, z\right) \leq d^k\left(y^A, y^B\right)$.*

Assumption 1 says that, in each state, there is at least one way to allocate resources that leaves both individuals strictly better off than if they were pursuing their outside options. An argument for this assumption that, when public goods are present, individuals should be able to coordinate on public expenditures, and thus achieve an outcome that is superior to that obtained from non-cooperative behaviour.

Assumption 2 says that the allocation given by the joint maximisation of current period utilities for a given pair of Pareto weights becomes more attractive relative to the current period outside option as one's own income declines or as the spouse's income increases, at least when the Pareto weights used yield a level of utility for oneself which is no higher than that obtained from the outside option. This assumption is consistent with the case where the outside option is given by a Nash equilibrium of the stage game; and it ensures that the participation constraint is tightened whenever an individual experiences an increase in income or his spouse experiences a decrease.

Assumption 3 imposes an upper bound on the marginal improvement in the outside option from a marginal increase in own income. Specifically, this improvement is assumed to be no larger than the marginal utility of private consumption in an efficient allocation of current resources that would cause his participation constraint to bind. In words, it means that when an individual brings home extra income, he can at best insist on spending all of it on private consumption (since the marginal gain from this extra income in the noncooperative outcome is assumed to be smaller). This is also consistent with the case where the outside option is given by a Nash equilibrium of the stage game.

Given this environment, we shall consider feasible 'agreements' within the household. Formally, an 'agreement' is a plan for allocating funds to each type of good in each period, (potentially) contingent on the current state and the history of past states. An agreement is feasible if neither individual has an incentive to opt for the outside option over the agreement after any possible history.

2.1 An Example with Memoryless Agreements

In this section, we analyse 'memoryless' agreements; i.e. agreements in which expenditures are contingent upon the current state of nature but independent of the history of past shocks. In particular, these agreements rule out the possibility that household members borrow from one another to cope with adverse shocks. This restriction is likely to be unrealistic: allowing for borrowing and lending in an environment with limited commitment should make Pareto improving reallocations possible. However, the restricted setting is ideal for demonstrating the effect, described in the introduction, that the presence of public goods has on the agreement.

Furthermore, it is assumed that $\frac{\partial d_A}{\partial y_B} = \frac{\partial d_B}{\partial y_A} = 0$; i.e. a change in the income of one's partner has no effect on one's own utility from the current period in autarky. This condition may be unrealistic given the presence of public goods on which one or both partners may spend money in a noncooperative equilibrium. However, the assumption simplifies the exposition. Both restrictions are relaxed in the following section, where the analysis is repeated in a more general setting.

A memoryless agreement can be written as $\left\{x_s^A, x_s^B, z_s\right\}_{s\in\mathcal{S}}$, where the expenditure levels x_s^A, x_s^B and z_s are prescribed in each period that the realised state of nature is s. If the environment is characterised by limited commitment, then such an agreement is feasible if and only if, for each state of nature, the continuation value each individual receives from the agreement is at least as large as that obtained from the outside option. We can write the feasibility constraints as follows:

$$\begin{aligned} u^i\left(x_s^i, z_s\right) + \frac{\beta}{1+\beta}\sum_{r\in\mathcal{S}}\pi_r u^i\left(x_r^i, z_r\right) &\geq d^i\left(y_s^i, y_s^{-i}\right) + \beta v_{aut}^i \\ \text{for each } s &\in \mathcal{S}, i \in \{A, B\} \end{aligned}$$

Let $M(v)$ be the maximum ex-ante utility that B can obtain from a memoryless agreement if the environment is characterised by limited comitment and the ex-ante utility received by A is at least v. Then $M(v)$ is given by

$$\max_{\{x_s^A, x_s^B, z_s\}} : \frac{1}{1+\beta}\sum_{r\in\mathcal{S}}\pi_r u^B\left(x_r^B, z_r\right) \tag{2}$$

subject to

$$\lambda : \frac{1}{1+\beta}\sum_{r\in S}\pi_r u^A\left(x_r^A, z_r\right) \geq v \tag{3}$$

and for each $s \in S$,

$$\theta_s^A : u^A\left(x_s^A, z_s\right) + \frac{\beta}{1+\beta}\sum_{r\in S}\pi_r u^A\left(x_r^A, z_r\right) \geq d^A\left(y_s^A, y_s^B\right) + \beta v_{aut}^A \tag{4}$$

$$\theta_s^B : u^B\left(x_s^B, z_s\right) + \frac{\beta}{1+\beta}\sum_{r\in S}\pi_r u^B\left(x_r^B, z_r\right) \geq d^B\left(y_s^A, y_s^B\right) + \beta v_{aut}^B \tag{5}$$

$$\tau_s : x_s^A + x_s^B + z_s \leq y_s^A + y_s^B \tag{6}$$

The condition in (3) ensures that person A receives an ex-ante utility of at least v. The conditions in (4) and (5) ensure that in each possible state, the utility levels A and B receive from the agreement are at least as large as that obtained from the outside option. The budget constraints, one for each state, are in (6). From the first-order conditions, we obtain

$$\begin{aligned} & \frac{1}{1+\beta}u_1^B(..) + \theta_s^B\left(1 + \frac{\beta\pi_s}{1+\beta}\right)u_1^B(..) \\ = & \frac{\lambda}{1+\beta}u_1^A(..) + \theta_s^A\left(1 + \frac{\beta\pi_s}{1+\beta}\right)u_1^A(..) \\ = & \frac{1}{1+\beta}\left[u_2^B(..) + \lambda u_2^A(..)\right] + \theta_s^B\left(1 + \frac{\beta\pi_s}{1+\beta}\right)u_2^B(..) + \theta_s^A\left(1 + \frac{\beta\pi_s}{1+\beta}\right)u_2^A(..) \end{aligned}$$

where $u_1^i(..)$ and $u_2^i(..)$ are the marginal utility to person i from the private and the public good respectively.

If the participation constraints do not bind for some state r, then $\theta_r^A = \theta_r^B = 0$ and

$$u_1^B(..) = \lambda u_1^A(..) = u_2^B(..) + \lambda u_2^B(..)$$

Thus, the first-order conditions are the same in each state that the participation constraint does not bind. It is straightforward to show that, for such states, the larger is aggregate income, the more is spent on each type of good: i.e. if states r and s are such that $y_s^A + y_s^B > y_r^A + y_r^B$, and $\theta_r^i = \theta_s^i = 0$ for $i = A, B$, then $x_s^A > x_r^A, x_s^B > x_r^B$ and $z_s > z_r$.

If person A's participation constraint binds, and that of person B is slack in state s, then $\theta_s^A > 0, \theta_s^B = 0$ and

$$u_1^B (..) = \lambda_s u_1^A (..) = u_2^B (..) + \lambda_s u_2^B (..)$$

where $\lambda_s = \lambda + \theta_s^A (1 + \beta + \beta \pi_s) > \lambda$.

Furthermore, if person A's participation constraint binds in state s, then using the fact that the condition in (4) is satisfied with equality, and the promised value to A from the agreement equals v in each period, we have

$$u^A \left(x_s^A, z_s\right) + \beta v = d^A \left(y_s^A, y_s^B\right) + \beta v_{aut}^A$$

Then the allocation of resources in state s prescribed by the agreement must correspond to the solution of the following problem:

$$\begin{aligned} \max_{x^A, x^B, z} &: \quad u^B \left(x^B, z\right) \qquad (7) \\ \text{subject to} &: \\ &: \quad u^A \left(x^A, z\right) \geq d^A \left(y^A, y^B\right) - \beta \left(v - v_{aut}^A\right) \\ &: \quad x^A + x^B + z \leq y^A + y^B \end{aligned}$$

when $y^A = y_s^A, y^B = y_s^B$.

If not, there must be a possible reallocation of resources in state s that satisfies person A's promise-keeping and participation constraints and gives person B a strictly higher continuation value, which is ruled out by definition.

The following lemma enables us to compare allocations across states in which person A's participation constraint is binding.

Lemma 1 *Let $x^A \left(y^A, y^B\right), x^B \left(y^A, y^B\right), z \left(y^A, y^B\right)$ be the solution to the maximisation problem described in (7). Then $x^B \left(y^A, y^B\right)$ and $z \left(y^A, y^B\right)$ are increasing in y^B; and $x^A \left(y^A, y^B\right)$ is decreasing in y^B.*

Suppose the states r and s are such that $y_r^B < y_s^B, y_r^A = y_s^A$. Then, using Lemma 1, $x^A \left(y_r^A, y_r^B\right) > x^A \left(y_s^A, y_s^B\right)$. If the participation constraint of person A is binding in both states, $x_r^A = x^A \left(y_r^A, y_r^B\right)$, $x_s^A = x^A \left(y_s^A, y_s^B\right) \implies x_r^A > x_s^A$. We have thus established the following proposition.

Proposition 1 *For an agreement that lies on the frontier of the set of feasible and memoryless agreements, if the states r and s are such that $y_r^B < y_s^B, y_r^A = y_s^A$, $\frac{\partial d_A}{\partial y_B} = 0$, and person A's participation constraint binds in both states, then his prescribed private expenditures are such that $x_r^A > x_s^A$.*

We have thus established the counter-intuitive result that, comparing two states where person A obtains the same income, and his participation constraint is binding, his private expenditures are *higher* in the state where person B has a *lower* income. Comparing across multiple states with varying levels of income for person B, and the same income for person A, it is clear from Assumption 2 that A's participation constraint will bind if his partner's income is sufficiently low. Then Proposition 1 implies that the non-monotonic effect of B's income on A's private expenditures comes into effect when the latter's participation constraint begins to bind; i.e. when B is sufficiently poor that A refuses to provide extra insurance. In the next section, it will be shown that a similar result obtains when we allow the allocation of expenditures to depend on the past history of shocks.

2.2 History-dependent Agreements

In this section, we analyse agreements where the allocation of expenditures can be contingent on the current state as well as on the past sequence of states. Let $\mathcal{H}_t$ be the set of all possible sequence of states $(s_1, s_2.., s_t)$ in period t. Then a history-dependent agreement can be described fully by a sequence of functions $\left\{x_t^A(.), x_t^B(.), z_t(.)\right\}_{t=1..\infty}$ where $x_t^A : \mathcal{H}_t \longrightarrow \mathcal{R}_+$ specificies the sum of money allocated to person A's private goods for each possible history in period t, and x_t^B and z_t are defined similarly.

For the agreement to be feasible in an environment with limited commitment, it must have the feature that at each date and at every contingency, each member receives as much utility from the agreement as they do from their outside option; i.e.

$$u^k\left(x_t^k(h_t), z_t(h_t)\right) + \beta E_t \sum_{\tau=1}^{\infty} \beta^{\tau-1} u_{t+\tau}^k\left(x^k(h_{t+\tau}), z_{t+\tau}(h_{t+\tau})\right) \geq d^k\left(y_s^A, y_s^B\right) + \beta v_{aut}^k$$

for each $h_t \in \mathcal{H}_t$, $k = A, B$, and $t = 0, 1, ..\infty$.

On the left-hand side, the first expression is the utility obtained from the agreement in period t if the realised history is h_t, and the second expression is the discounted expected future utility from the agreement. The right-hand side is the value of the outside option in the current period, assuming that the realised state in period t is s.

In the case of history-dependent agreements, each possible history in each period yields a distinct constraint for each individual. As the set of possible histories grows exponentially with t, it is not possible to adopt the approach used in the previous section to analyse such agreements. Fortunately, the problem is very similar to that considered in Kocherlakota (1996) and Ligon, Thomas and Worrall (2002), and the same recursive approach is applicable here. Define $P(v)$ as the maximum ex-ante utility that person B can obtain from a history-dependent agreement if person A must receive a utility of at least v. Then $P(v)$ satisfies the following Bellman equation:

$$P(v) = \max_{\{x_s^A, x_s^B, z_s, w_s\}} \sum_{s=1}^{N} \left[\pi_s u^B \left(x_s^B, z_s\right) + \beta P\left(w_s\right)\right] \tag{8}$$

subject to

$$\lambda : \sum_{s=1}^{N} \left[\pi_s u^A \left(x_s^A, z_s\right) + \beta w_s\right] \geq v \tag{9}$$

and for each $s \in S$,

$$\theta_s^A \quad : \quad u^A \left(x_s^A, z_s\right) + \beta w_s \geq d^A \left(y_s^A, y_s^B\right) + \beta v_{aut}^A \tag{10}$$

$$\theta_s^B \quad : \quad u_B \left(x_s^B, z_s\right) + \beta P\left(w_s\right) \geq d^B \left(y_s^A, y_s^B\right) + \beta v_{aut}^B \tag{11}$$

$$\tau_s \quad : \quad x_s^A + x_s^B + z_s \leq y_s^A + y_s^B \tag{12}$$

The formulation above reduces the problem of finding a complete contingency plan which is constrained efficient and awards person A an ex-ante utility of v to the much simpler task of choosing, for each possible state in the current period, an allocation of expenditures x_s^A, x_s^B, z_s and the ex-ante utility (or promised value) w_s with which A would enter the following period. The expression in the maximand is, by definition, the maximum possible utility that B can obtain from such an allocation. The condition in (9) ensures that the allocation leaves A with an ex-ante utility of at least v. The conditions in (10) and (11) ensure that, in each state, A and B receive as much utility from the agreement as they would from their outside options. The budget constraints are in (12).

From the first-order conditions to the problem and the envelope condition, we obtain, for each $s \in S$,

$$P'\left(w_s\right) = \frac{\pi_s P'\left(v\right) - \theta_s^A}{\pi_s + \theta_s^B} \tag{13}$$

and

$$u_1^B\left(..\right) = \lambda_s u_1^A\left(..\right) = u_2^B\left(..\right) + \lambda_s u_2^A\left(..\right) \tag{14}$$

where $\lambda_s = -P'\left(w_s\right)$.

Note that the conditions in (14) for state s are equivalent to the first-order conditions of the following maximisation problem:

$$\max_{x^A, x^B, z} \lambda u^A \left(x^A, z\right) + u^B \left(x^B, z\right) \tag{15}$$

$$\text{subject to:} \qquad x^A + x^B + z \leq y^A + y^B$$

when $y^A = y_s^A$, $y^B = y_s^B$, and $\lambda = \lambda_s$. Therefore, λ_s is the weight given to A's preferences in state s in allocating resources among the three types of goods; and the current utility obtained by A and B can be written as $\mu^A \left(y_s^A, y_s^B, \lambda_s\right)$ and $\mu^B \left(y_s^A, y_s^B, \lambda_s\right)$ respectively. Thus, the allocation of resources and the promised future utilities can be fully described by the pair (λ_s, w_s).

To see how (λ_s, w_s) varies across states for a given v, it is convenient to group the states according to the following three categories:

(i) Let $S^A(v)$ be the set of states for which the pair (λ, v) violates person A's participation constraint; i.e.

$$S^A(v) = \left\{ s \in S : \mu^A\left(y_s^A, y_s^B, \lambda\right) + \beta v < d^A\left(y_s^A, y_s^B\right) + \beta v_{aut}^A \right\}$$

where $\lambda = -P'(v)$.

(ii) Similarly, let $S^B(v)$ be the set of states for which the pair (λ, v) violates B's participation constraint:

$$S^B(v) = \left\{ s \in S : \mu^B\left(y_s^A, y_s^B, \lambda\right) + \beta P(v) < d^B\left(y_s^A, y_s^B\right) + \beta v_{aut}^B \right\}$$

(iii) Let $S^0(v)$ be the set of states for which the pair (λ, v) satisfies the constaints of both individuals:

$$S^0(v) = \left\{ \begin{array}{c} s \in S : \mu^A\left(y_s^A, y_s^B, \lambda\right) + \beta v \geq d^A\left(y_s^A, y_s^B\right) + \beta v_{aut}^A \\ \text{and } \mu^B\left(y_s^A, y_s^B, \lambda\right) + \beta P(v) \geq d^B\left(y_s^A, y_s^B\right) + \beta v_{aut}^B \end{array} \right\}$$

The three categories exhaust all possibilities. Further, because of Assumption 1, the first two categories are mutually exclusive. Then, following the reasoning of Proposition 1 in Ligion, Thomas and Worrall (2002), it is straightforward to establish the following result.

Lemma 2 *If, in a constrained efficient agreement, person A is awarded a utility of v, then (i) for $s \in S^A(v)$, $\theta_s^A > 0, \theta_s^B = 0$ and $\lambda_s > \lambda, w_s > v$; (ii) for $s \in S^B(v)$, $\theta_s^A = 0, \theta_s^B > 0$ and $\lambda_s < \lambda, w_s < v$; and (iii) for $s \in S^0(v)$, $\theta_s^A = \theta_s^B = 0$ and $\lambda_s = \lambda, w_s = v$.*

The lemma describes how, in a constrained efficient agreement, the allocation of resources and continuation values is determined for each possible state in the first period, depending on the parameter v, the target utility for person A in the agreement.

Specifically, the following procedure is used. For each state, one checks if the pair $(-P'(v), v)$ satisfies the participation constraints of both individuals (where $-P'(v)$ is the ratio of pareto weights for allocating resources in that period and v is the continuation value for person A). If so, then this pair is used; i.e. $\lambda_s = -P'(v)$, $w_s = v$. If not, then v is adjusted till the constraint that was initially violated is just binding.

Then, to find the allocation of resources and continuation values in the second period, the procedure is repeated using w_s – the continuation value in the realised state in period 1 – as the target utility for person A. And so on for the subsequent periods. The reason that the procedure can be applied iteratively is that, as explained in Ligon, Thomas and Worrall (2002), in any constrained efficient agreement, all continuation agreements must also be constrained efficient. If not, it is possible to improve on the original agreement by replacing a continuation agreement with another that is constrained efficient.

The dynamics of the constained efficient agreement, as discussed thus far, corresponds exactly to the case where all expenditures are private. Proposition 1 in Ligon, Thomas and Worrall (2002) which describes how the ratio of pareto weights in the constrained efficient agreement evolves over time according to the realised history, applies, *mutatis mutandis*, to the present setting as well. However, the results diverge when one considers how the *level* of expenditures on private goods evolve in the absence and presence of public goods.

To this end we begin by comparing levels of expenditures across different states following some history, when the continuation value for person A equals v. Comparing expenditures across states which belong to $S^0(v)$ is straightforward. As the weights used for A's and B's preferences to allocate resources are the same in all these states, expenditure on each type of good is increasing in aggregate income: if $r, s \in S^0(v)$, and $y_r^A + y_r^B < y_s^A + y_s^A$ then $x_r^A < x_s^A$, $x_r^B < x_s^B$, and $z_r < z_s$.

To compare expenditures across states which belong to $\mathcal{S}^A(v)$ (and, similarly, $\mathcal{S}^B(v)$) we proceed as follows. If $s \in \mathcal{S}^A(v)$, the allocation in that state is given by the solution to the following maximisation problem:

$$\begin{aligned} \max_{x^A, x^B, z, w} &: \quad u^B\left(x^B, z\right) + \beta P(w) \\ \text{s.t.} &: \\ \lambda &: \quad u^A\left(x^A, z\right) = d^A\left(y^A, y^B\right) - \beta\left(w - v_{aut}^A\right) \\ \tau &: \quad x^A + x^B + z \leq y^A + y^B \end{aligned} \tag{16}$$

with $y^A = y_s^A$, $y^B = y_s^B$.

To see this, note, first, that the solution to (16) yields an allocation that is feasible: by construction, it satisfies person A's participation constraints; and it must also satisfy person B's participation constraints given the assumption that there is at least one allocation in each period that leaves both individuals better off than in autarky. Furthermore, an allocation in a constrained efficient agreement must solve the maximisation problem in (16). If not, there is a possible reallocation that leaves person A at least as well-off and provides strictly higher utility to person B.

Note that the solution to (16) depends on the current state of nature s but is independant of v, the promised value in the preceding period. This means that when the participation constraint binds for either individual at a point in time, we ignore the past history of states, and the allocation in that period is contingent only on the current state. Therefore, if person A has a binding participation constraint in some state s after any history, we can denote this allocation by $x_s^A[A], x_s^B[A], z_s[A], w_s[A]$ (and similarly for person B).

Compared to the maximisation problem in (7), which related to 'memoryless' agreements, there is an additional choice variable, w, in (16). In the former case, the promised value to each individual remained fixed over time, as we were considering a type of agreement that ignored the past sequence of events. In the latter case, the promised values can change over time, and serves as an additional mechanism to ensure that the participation constraints of each individual are satisfied in each period. Intuitively, this additional mechanism should dampen the effect – of raising private expenditures at the expense of public expenditures in order to satisfy participation constraints – noted in the previous section.

In particular, we obtain the following equivalent of Lemma 1:

Lemma 3 *: Let* $\left\{x^A\left(y^A, y^B\right), x^B\left(y^A, y^B\right), z\left(y^A, y^B\right), w\left(y^A, y^B\right)\right\}$ *denote the solution to the maximisation problem in (16). Then*

(i) $\frac{\partial x^B\left(y^A, y^B\right)}{\partial y^B} > 0$, $\frac{\partial z\left(y^A, y^B\right)}{\partial y^B} > 0$ *for* $\frac{d^A}{y^B}$ *sufficiently small;*

(iii) $\frac{\partial x^A\left(y^A, y^B\right)}{\partial y^B} \geq 0$ *iff*

$$\frac{\partial \bar{x}^A}{\partial y^B} + \frac{\partial \bar{x}^A}{\partial \lambda}\left[\frac{\frac{\partial d^A}{\partial y^B} - u_2^A\left(..\right)\frac{\partial \bar{z}}{\partial y^B}}{u_2^A\left(..\right)\frac{\partial \bar{z}}{\partial \lambda} + \beta\frac{\partial \bar{w}}{\partial \lambda}}\right] \geq 0 \tag{17}$$

at $\lambda = \sigma\left(y^A, y^B\right)$*, where* $\bar{x}^A\left(y^A, y^B, \lambda\right)$*,* $\bar{x}^B\left(y^A, y^B, \lambda\right)$*,* $\bar{z}\left(y^A, y^B, \lambda\right)$*,* $\bar{w}\left(y^A, y^B, \lambda\right)$ *is the solution to problem*

$$\max_{x^A, x^B, z, w} \lambda u^A\left(x^A, z\right) + u^B\left(x^B, z\right) + \beta\left(w + P\left(w\right)\right) \tag{18}$$
$$s.t. \quad : \quad x^A + x^B + z \leq y^A + y^B$$

and $\sigma\left(y^A, y^B\right)$ *is the value of the Lagrange multiplier to the problem in (16).*

The lemma states that for $\frac{d^A}{y^B}$ close to zero – i.e. the change in the outside option of person A for changes in the income of person B is sufficiently small – x^B and z are increasing in y^B in the solution to (16). Consequently, comparing two states of nature where person A's participation constraint is binding after some history, and holding y^A constant, expenditures on x^B and z should be higher in the state where y^B is higher, for $i \in \{A, B\}$.

Also, x^A is increasing in y^B if and only if the condition in (17) is satisfied. This condition should be interpreted as follows. The first term on the left-hand side is the change in x^A obtained from an increase in y^B were the pareto weights from the preceding period retained. The second term is the change in x^A from the decrease in person A's pareto weight that is necessary to ensure that his participation constraint is just satisfied following a marginal increase in y^B. If the net effect of these two changes is negative, then expenditures on person A's private expenditures are higher in the state that y^B is lower.

Note, first, that in the absence of public goods, we must have $\frac{\partial \bar{z}}{\partial y^B} = 0$. Then the expression in the square brackets is positive, and therefore x^A is increasing in y^B. Thus, we obtain the result that when all expenditures are private, person A's private consumption is (weakly) increasing in person B's income. On the other hand, if public goods expenditures are sensitive to changes in income, then a smaller y^B negatively affects person A's well-being, which – if larger than the negative effect on person A's outside option – must be offset by a larger pareto weight for A.

How much larger the pareto weight must be depends on the terms in the denominator within the square brackets in (17). The first term which will be close to zero if the two agents have similar preferences regarding the public good. The second term, $\beta\frac{\partial\bar{w}}{\partial\lambda}$, represents the increase in person A's future promised utility following a marginal increase in his pareto weight. It is shown in Lemma 6 in the appendix that $\frac{\partial\bar{w}}{\partial\lambda}$ is decreasing in β. Therefore, given any y^A, y^B and λ, the condition in (17) is violated for β sufficiently small.

Thus, using Lemma 3, we obtain the following result.

Proposition 2 *If state s is such that* $y_r^A = y_s^A$, $y_r^B < y_s^B$, *then, for* $\frac{\partial d^A}{\partial y^B}$ *sufficiently small, we have* $z_r[A] < z_s[A]$, *and* $x_r^B[A] < x_s^B[A]$. *Furthermore, there exists a* $\beta^* > 0$ *such that for* $\beta < \beta^*$, $x_r^A[A] > x_s^A[A]$.

See Appendix for proof.

Proposition 2 states that, in comparing allocations across states in which person A's participation constraint is binding, and holding person A's income fixed, more is spent on the public good and person B's private expenditures in the state that aggregate income is higher. On the other hand, if the discount factor is sufficiently small, then person A's private expenditures are *smaller* in the state that aggregate income is higher. As in the case with 'memoryless' agreements, this last result implies a non-monotonic relationship between person B's income and person A's private expenditures, with the non-monotonicity coming into effect when person B's current income is so low that her partner's participation constraint begins to bind.

Note that there is no β sufficiently large to guarantee that person A's private expenditures are larger in the state that aggregate income is higher. However, if β is sufficiently close to 1, then the first-best contract can be implemented, in which the participation constraints never bind; and therefore the 'perverse insurance' effect does not occur.

Intuitively, the effect of a small β is as follows. Comparing two states in which person A has the same income and his participation constraint is binding, he must receive a larger compensation in the state where aggregate income, and therefore expenditures on public goods, are lower. In the constrained efficient agreement, part of this compensation will take the form of 'borrowing' and 'lending' between the agents, specifically a shift in the promised utilities w and $P(w)$. However, the lack of commitment also imposes restrictions on the extent to which individuals can 'borrow' from one another. It may not be possible to promise a high level of future consumption to one individual as this makes it tempting for the partner to renege on the agreement. In particular, if the individuals are very impatient, (i.e. β is low), they do not value compensation that is received in the future. In addition, this impatience limits the amount that one can promise to 'pay back'. Thus, when impatience is high, we are more likely to observe situations where compensation takes the form of increased private expenditures in the current period.

As noted in Proposition 2, for the 'perverse' insurance effect to take place, we also require that person A's outside option is not significantly worse in the state where person B has a smaller income; i.e. $\frac{\partial d^A}{\partial y^B}$ is required to be small. This is because, if person A's outside option is significantly less attractive in the state that B is poorer, it would not be necessary to 'bribe' A to tempt him to remain in the agreement in this state.

This comparison of allocations across states translates easily into a comparison of allocations over time. Suppose in period t, the realised state is s and this is followed in period $t+1$ by state r with $y_r^A = y_s^A$, $y_r^B < y_s^B$. It can be shown that if the participation constraint of person A is binding in period t, it must bind again in period $t+1$ (see Proposition 4 in the appendix). Therefore, his private expenditures is $x_s^A[A]$ in period t followed by $x_r^A[A]$ in period $t+1$. Thus, we have the following corollary to Proposition 2.

Corollary 1 *If the participation constraint is binding for A in state s, period t and the state of nature, r in period t+1 is such that* $y_r^A = y_s^A$, $y_r^B < y_s^B$, *then for* $\frac{\partial d^A}{\partial y^B}$ *sufficiently smalll,* $x^B(t+1) < x^B(t)$, $z(t+1) < z(t)$ *and* $\exists \beta^* > 0$ *s.t. for* $\beta < \beta^*$, $x^A(t+1) > x^A(t)$.

The corresponding set of results are, of course, obtained with the same reasoning when person B's participation constraint is binding.

We have thus shown that, within an insurance agreement, when agents are sufficiently impatient, a negative shock to the income of one agent can lead to an increase in the private consumption of another, under limited commitment and the presence of public expenditures. In particular, this means that the co-movement of private consumption of agents is not a necessary condition for insurance within a group.

3 Conclusion

Theories of limited commitment were originally developed to provide a better understanding of interactions between households; and in particular to explain why informal arrangements among households may not lead to full insurance. These theories may now serve to develop a better understanding of intra-household allocation; and to account for the growing evidence of inefficiency in the allocation of expenditures within the household.

This paper makes the point that, in importing these theories for developing models of intra-household dynamics, we ought not to ignore the fact that public goods constitute a very significant part of household expenditures[6]. In the standard model of limited commitment (where public goods are absent), individual consumption covaries with aggregate consumption; but this relationship can break down if public goods are present.

[6] The fact is well-recognised in the literature on intra-household allocation, and static (one-period) models of the household in the literature have, in general, explicitly allowed for the possibility of household public goods. See, for example, the seminal work of Becker (1981), as well as Manser and Brown (1980), McElroy and Horney (1981), Lundberg and Pollak (1993), Browning and Chiappori (1998) and Chen and Woolley (2001).

Therefore, co-movement in the private consumption of household members should not be regarded as a necessary condition for (partial) insurance within the household. The model used in this paper illustrates that divergence of (private) consumption can take place within a cooperative agreement. In the presence of commitment problems and public goods, such divergence may be the efficient way to deal with adverse shocks.

In the recent past, household surveys in developing countries have begun to collect information on individual well-being; for example, measures of the nutritional status of household members. While this data can provide valuable insights about decision-making and bargaining power within the household, it is evident that measures of individual well-being present only a partial picture of household dynamics. The analysis provided in this paper highlights the importance of analysing this data within a conceptual framework that explicitly accounts for the presence of public goods within the household. In particular, the absence of co-movement in private consumption, by itself, should not be interpreted as the absence of intra-household insurance, as some of the empirical literature has tended to do.

4 Appendix

Proof. of Lemma 1: Define $g(z)$ implicitly using the following identity:

$$u^A(g(z), z) \equiv K$$

Differentiating the identity above w.r.t. z, we obtain

$$u_1^A g'(z) + u_2^A \equiv 0$$

$$\implies g'(z) = -u_2^A / u_1^A$$

Differentiating again, we obtain

$$\left[u_{11}^A g'(z) + u_{12}^A\right] g'(z) + u_1^A g''(z) + u_{21}^A g'(z) + u_{22}^A \equiv 0$$

$$\implies g''(z) = -\frac{\left[u_{11}^A g'(z) + 2u_{12}^A\right] g'(z) + u_{22}^A}{u_1^A} > 0$$

Further define $h(z) = g(z) + z$. Note that the solution to the problem in (7), must satisfy the condition $u_2^A < u_1^A$ (if not, transferring expenditures from x^B to z would slacken the promise keeping constraint as well as improve person B's utility). For allocations that satisfy this condition, we have $g'(z) > -1$

$$\implies h'(z) = \frac{d}{dz}[g(z) + z] > 0$$

Since $h(.)$ is monotonic increasing in z, it is invertible. Then, we can define $v\left(x^B, \xi\right)$ in the following manner:

$$v\left(x^B, \xi\right) = u^B\left(x^B, h^{-1}(\xi)\right)$$

Therefore, the optimisation problem in (7) is equivalent to

$$\begin{aligned} \max_{x^B,\xi} &: \quad v\left(x^B,\xi\right) \\ \text{subject to} &: \quad x^B+\xi \leq y^A+y^B \end{aligned} \tag{19}$$

when $K = d^A\left(y^A, y^B\right) - \beta\left(v - v^A_{aut}\right)$ and $\xi = g(z) + z$. Proposition 1 in Quah (2007) provides sufficient conditions under which x^B, ξ are increasing in y in the solution to (19). In particular it suffices that $v\left(x^B,\xi\right)$ is supermodular and concave. To verify these conditions are satisfied, we compute all the first- and second-order derivatives of $v(.)$ as follows:

$$\begin{aligned} v_1 &= u^B_1 > 0 \\ v_2 &= u^B_2 \frac{\partial h^{-1}}{\partial \xi} \\ &= u^B_2 \frac{1}{h'(z)} \text{ at } z = h^{-1}(\xi) \\ &= u^B_2 \frac{1}{g'(z)+1} > 0 \\ v_{11} &= u^B_{11} < 0 \\ v_{22} &= u^B_{22} \frac{1}{h'(z)} + u^B_2 \left[-\frac{h''(z)}{[h'(z)]^2} \right] \\ &= u^B_{22} \frac{1}{h'(z)} - u^B_2 \frac{g''(z)}{[h'(z)]^2} < 0 \\ v_{12} &= u^B_{12} \frac{1}{h'(z)} > 0 \text{ since } u^B_{12}(.) > 0 \end{aligned}$$

Thus, given our assumption that $u^B(.)$ is supermodular and concave, we obtain that $v(.)$ is also supermodular and concave. Therefore, applying Proposition 1 in Quah (2007), we obtain the result that x^B, ξ are increasing in y. Now, since the relationship between ξ and z is monotonic, z is increasing in y. Therefore x^A is decreasing in y^B. ■

Proof. of Lemma 2: (i) For $s \in S^A(v)$, suppose $\theta^A_s = 0$. Then, using equation (13), $P'(w_s) \geq P'(v) \iff w_s \leq v$; and since $\lambda_s = -P'(w_s)$, we obtain $\lambda_s \leq \lambda$. Then,

$$\begin{aligned} \mu^A\left(y^A_s, y^B_s, \lambda_s\right) + \beta w_s &\leq \mu^A\left(y^A_s, y^B_s, \lambda\right) + \beta v \\ &< d^A\left(y^A_s, y^B_s\right) + \beta v^A_{aut} \end{aligned}$$

i.e. person A's participation constraint would be violated in state s. Therefore, we must have $\theta^A_s > 0$. This implies that the constraint is binding with equality:

$$\mu^A\left(y^A_s, y^B_s, \lambda_s\right) + \beta w_s = d^A\left(y^A_s, y^B_s\right) + \beta v^A_{aut}$$

Then, using Assumption 1, we obtain

$$\mu^B\left(y_s^A, y_s^B, \lambda_s\right) + \beta P\left(w_s\right) > d^B\left(y_s^A, y_s^B\right) + \beta v_{aut}^B$$

$$\implies \theta_s^B = 0$$

Then, using equation (13), we obtain $w_s > v, \lambda_s > \lambda$.

(ii) The argument used for part (i) can be used here as well. If $s \in S^B(v)$ and $\theta_s^B = 0$, then person B's participation constraint would be violated. Therefore, we obtain $\theta_s^B > 0$ and $\theta_s^A = 0 \implies w_s < v, \lambda_s < \lambda$.

(iii) For $s \in S^0(v)$, suppose $\theta_s^A > 0$. Then, person A's participation constraint binds with equality in state s:

$$\mu^A\left(y_s^A, y_s^B, \lambda_s\right) + \beta w_s = d^A\left(y_s^A, y_s^B\right) + \beta v_{aut}^A$$

and using Assumption 1, we obtain

$$\mu^B\left(y_s^A, y_s^B, \lambda_s\right) + \beta P\left(w_s\right) > d^B\left(y_s^A, y_s^B\right) + \beta v_{aut}^B$$

$$\implies \theta_s^B = 0$$

Then, using equation (13), we obtain $w_s > v$ and $\lambda_s > \lambda$. This implies

$$\mu^A\left(y_s^A, y_s^B, \lambda\right) + \beta v < d^A\left(y_s^A, y_s^B\right) + \beta v_{aut}^A$$

which contradicts the definition of $S^0(v)$. Therefore, we must have $\theta_s^A = 0$. Similarly, we can show that for $s \in S^0(v)$, we must have $\theta_s^B = 0$. Then, using equation (13), we obtain $w_s = v \iff \lambda_s = \lambda$. ■

The proof of Proposition 2 uses the following lemma.

Proof. of Lemma 3: (i) To obtain the first result, we shall use Theorem 2 in Quah (2007) which allows comparative statics when the constraint set is varying in the parameter involved. In addition, we first consider the case where only the budget constraint varies with y^B and the participation constraint of person A is fixed at some constant $\tilde{K}$. Define $Q(.) = P^{-1}(.)$. Then the problem in (16) can be rewritten as

$$\max_{x^A, x^B, z, w^B} : u^B\left(x^B, z\right) + \beta w^B \tag{20}$$

subject to :

$$: x^A + x^B + z \leq y^A + y^B \tag{BC}$$

$$: u^A\left(x^A, z\right) + \beta Q\left(w^B\right) = \tilde{K} \tag{PC}$$

Define $\tilde{g}\left(z, w^B\right)$ implicitly according to the following identity:

$$u^A\left(\tilde{g}\left(z, w^B\right), z\right) + \beta Q\left(w^B\right) \equiv \tilde{K}$$

Then, the budget constraint can be written as

$$\tilde{g}\left(z, w^B\right) + x^B + z \leq y$$

First, we compute the first- and second-order derivatives of $\tilde{g}(.)$ using the Implicit Function theorem:

$$\begin{aligned}
\tilde{g}_1 &= -\frac{u_2^A}{u_1^A} < 0 \\
\tilde{g}_2 &= -\frac{\beta Q'\left(w^B\right)}{u_1^A} > 0 \\
\tilde{g}_{12} &= \frac{\beta Q'\left(w^B\right)\left(u_{12}^A - u_2^A u_{11}^A / u_1^A\right)}{\left[u_1^A\right]^2} < 0 \\
\tilde{g}_{11} &= \frac{-u_{11}^A\left(\tilde{g}_1\right)^2 - 2u_{12}^A \tilde{g}_1 - u_{22}^A}{u_1^A} > 0 \\
\tilde{g}_{22} &= \frac{-u_{11}^A\left(\tilde{g}_2\right)^2 - \beta Q'\left(w^B\right)}{u_1^A} > 0
\end{aligned}$$

To apply Theorem 2 in Quah (2007), we first need to show that the set $S(y) = \left\{x^B, z, w^B : \tilde{g}\left(z, w^B\right) + x^B + z \leq y\right\}$ is increasing in the $C-$flexible set order. With this aim, define $D\left(x^B, z, w^B\right) = \tilde{g}\left(z, w^B\right) + x^B + z - y$. Then $S(y) = D^{-1}((-\infty, 0])$. And $S(y') = D^{-1}((-\infty, y' - y)$. Now, $D(.)$ is submodular and convex. Therefore, using Proposition 2 in Quah (2007), it is $C-$ submodular. Therefore, we have, using Proposition 4 in Quah (2007), that $D^{-1}((-\infty, k])$ is increasing in k in the sense of the $C-$flexible set order. Therefore, we obtain

$$S(y') \geq_i S(y) \text{ for } i = x^B, z, w^B$$

We can then apply Theorem 2 in Quah (2007) to establish that x^B, z and w^B in the solution to the revised problem in (20), increases in y^B. Then, by continuity, x^B, z and w^B also increase in y^B for $\frac{d^A}{y^B}$ sufficiently small when we replace $\tilde{K}$ by $d^A\left(y^A, y^B\right) + \beta v_{aut}^A$.

(ii) It is straightforward to show that $x^A\left(y^A, y^B\right)$ can be written as $\bar{x}^A(y^A, y^B, \sigma\left(y^A, y^B\right))$, and that corresponding expressions can be used for $x^B\left(y^A, y^B\right), z\left(y^A, y^B\right)$ and $w\left(y^A, y^B\right)$. Furthermore, as the allocation $x^A\left(y^A, y^B\right)$, $x^B\left(y^A, y^B\right)$, $z\left(y^A, y^B\right)$, $w\left(y^A, y^B\right)$ by definition satisfies the participation constraint of person A with equality we have

$$u^A\left(\bar{x}^A(y^A, y^B, \sigma\left(y^A, y^B\right)), \bar{z}(y^A, y^B, \sigma\left(y^A, y^B\right))\right) + \beta\bar{w}(y^A, y^B, \sigma\left(y^A, y^B\right)) = d^A\left(y^A, y^B\right) + \beta v_{aut}^A$$

$$\Longrightarrow \mu^A\left(y^A, y^B, \sigma\left(y^A, y^B\right)\right) + \beta\bar{w}(y^A, y^B, \sigma\left(y^A, y^B\right)) = d^A\left(y^A, y^B\right) + \beta v_{aut}^A \tag{21}$$

where $\mu^A(..)$ is as defined in section 2. Taking the derivative throughout w.r.t. y^A in (21) and rearranging, we obtain

$$\frac{\partial\sigma}{\partial y^A} = \frac{\left[\frac{\partial d^A}{\partial y^A} - \frac{\partial\mu^A}{\partial y^A}\right]}{\left[\frac{\partial\mu^A}{\partial\sigma} + \beta\frac{\partial\bar{w}}{\partial\sigma}\right]} \tag{22}$$

By Assumption (ii) $\frac{\partial d^A}{\partial y^A} - \frac{\partial \mu^A}{\partial y^A} \geq 0$, and likewise, the denominator within the square brackets on the right-hand side is greater than zero. Taking the derivative throughout the identity $x^A\left(y^A, y^B\right) = \bar{x}^A(y^A, y^B, \sigma\left(y^A, y^B\right))$ w.r.t. y^B, we obtain

$$\frac{\partial x^A}{\partial y^B} = \frac{\partial \bar{x}^A}{\partial y^B} + \frac{\partial \bar{x}^A}{\partial \sigma} \frac{\left[\frac{\partial d^A}{\partial y^B} - \frac{\partial \mu^A}{\partial y^B}\right]}{\left[\frac{\partial \mu^A}{\partial \sigma} + \beta \frac{\partial \bar{w}}{\partial \sigma}\right]}$$

Therefore, $\frac{\partial x^A}{\partial y^B} \geq 0$ iff

$$\frac{\partial \bar{x}^A}{\partial y^B} + \frac{\partial \bar{x}^A}{\partial \sigma} \frac{\left[\frac{\partial d^A}{\partial y^B} - \frac{\partial \mu^A}{\partial y^B}\right]}{\left[\frac{\partial \mu^A}{\partial \sigma} + \beta \frac{\partial \bar{w}}{\partial \sigma}\right]} \geq 0 \tag{23}$$

From the definition of $\mu^A\left(y^A, y^B, \lambda\right)$, we have $\frac{\partial \mu^A}{\partial y^B} \equiv u_1^A(..) \frac{\partial \bar{x}^A}{\partial y^B} + u_2^A(..) \frac{\partial \bar{z}}{\partial y^B}$ and $\frac{\partial \mu^A}{\partial \lambda} \equiv u_1^A(..) \frac{\partial \bar{x}^A}{\partial \lambda} + u_2^A(..) \frac{\partial \bar{z}}{\partial \lambda}$. Substituting for $\frac{\partial \mu^A}{\partial y^B}$ and $\frac{\partial \mu^A}{\partial \lambda}$ in (23) using these expressions and simplifying, we obtain the inequality in (17). ■

The proof of Proposition 2 makes use of the following lemma.

Lemma 4 *: Let $\bar{x}^A\left(y^A, y^B, \right)$, $\bar{x}^B\left(y^A, y^B, \right)$, and $\bar{z}\left(y^A, y^B, \right)$ be as in the statement of Lemma 3. Then*

$$\frac{\partial \bar{x}^A}{\partial \lambda}\frac{\partial \bar{z}}{\partial y^B} - \frac{\partial \bar{x}^A}{\partial y^B}\frac{\partial \bar{z}}{\partial \lambda} > 0$$

for all values of y^A, y^B and λ.

Proof. of Lemma 4: From the first-order conditions to the maximisation problem in (18),

$$\lambda u_1^A\left(\bar{x}^A\left(y^A, y^B, \lambda\right), \bar{z}\left(y^A, y^B, \lambda\right)\right) \equiv \lambda u_2^A\left(\bar{x}^A\left(y^A, y^B, \lambda\right), \bar{z}\left(y^A, y^B, \lambda\right)\right) + u_2^B\left(\bar{x}^B\left(y^A, y^B, \lambda\right), \bar{z}\left(y^A, y^B, \lambda\right)\right) \tag{24}$$

The terms $\bar{x}^A\left(y^A, y^B, \lambda\right)$, $\bar{x}^B\left(y^A, y^B, \lambda\right)$, and $\bar{z}\left(y^A, y^B, \lambda\right)$ must also satisfy the budget constraint:

$$\bar{x}^A\left(y^A, y^B, \lambda\right) + \bar{x}^B\left(y^A, y^B, \lambda\right) + \bar{z}\left(y^A, y^B, \lambda\right) = y^A + y^B \tag{25}$$

Derivating throughout (24) w.r.t. y^B and λ and rearranging, we obtain

$$\lambda\left(u_{11}^A - u_{12}^A\right)\frac{\partial \bar{x}^A}{\partial y^B} \equiv \lambda\left(u_{22}^A - u_{12}^A\right)\frac{\partial \bar{z}}{\partial y^B} + u_{12}^B \frac{\partial \bar{x}^B}{\partial y^B} + u_{22}^B \frac{\partial \bar{z}}{\partial y^B} \tag{26}$$

$$\lambda\left(u_{11}^A - u_{12}^A\right)\frac{\partial \bar{x}^A}{\partial \lambda} + u_1^A \equiv \lambda\left(u_{22}^A - u_{12}^A\right)\frac{\partial \bar{z}}{\partial \lambda} + u_{12}^B \frac{\partial \bar{x}^B}{\partial \lambda} + u_{22}^B \frac{\partial \bar{z}}{\partial \lambda} + u_2^A \tag{27}$$

Similarly, derivating throughout (25) w.r.t. y^B and λ we obtain

$$\frac{\partial \bar{x}^B}{\partial y^B} = 1 - \frac{\partial \bar{x}^A}{\partial y^B} - \frac{\partial \bar{z}}{\partial y^B} \tag{28}$$

$$\frac{\partial \bar{x}^B}{\partial \lambda} = -\frac{\partial \bar{x}^A}{\partial \lambda} - \frac{\partial \bar{z}}{\partial \lambda} \tag{29}$$

Substituting in (26) and (27) using (28) and (29) and rearranging,

$$\left[\lambda\left(u_{11}^{A}-u_{12}^{A}\right)+u_{12}^{B}\right]\frac{\partial\bar{x}^{A}}{\partial y^{B}} \equiv \left[\lambda\left(u_{22}^{A}-u_{12}^{A}\right)-u_{12}^{B}+u_{22}^{B}\right]\frac{\partial\bar{z}}{\partial y^{B}}+u_{12}^{B} \tag{30}$$

$$\left[\lambda\left(u_{11}^{A}-u_{12}^{A}\right)+u_{12}^{B}\right]\frac{\partial\bar{x}^{A}}{\partial\lambda} \equiv \left[\lambda\left(u_{22}^{A}-u_{12}^{A}\right)-u_{12}^{B}+u_{22}^{B}\right]\frac{\partial\bar{z}}{\partial\lambda}+u_{2}^{A}-u_{1}^{A} \tag{31}$$

Dividing (30) by (31) and rearranging,

$$\left[\lambda\left(u_{22}^{A}-u_{12}^{A}\right)-u_{12}^{B}+u_{22}^{B}\right]\left(\frac{\partial\bar{x}^{A}}{\partial\lambda}\frac{\partial\bar{z}}{\partial y^{B}}-\frac{\partial\bar{x}^{A}}{\partial y^{B}}\frac{\partial\bar{z}}{\partial\lambda}\right)\equiv\left(\frac{\partial\bar{x}^{A}}{\partial y^{B}}\left(u_{2}^{A}-u_{1}^{A}\right)-\frac{\partial\bar{x}^{A}}{\partial\lambda}u_{12}^{B}\right)$$

The first term within square brackets on the left-hand side, and the term on the right-hand side, are both negative. Therefore,

$$\frac{\partial\bar{x}^{A}}{\partial\lambda}\frac{\partial\bar{z}}{\partial y^{B}}-\frac{\partial\bar{x}^{A}}{\partial y^{B}}\frac{\partial\bar{z}}{\partial\lambda}>0$$

■

The proof of Lemma 5 makes use of the following proposition which obtains, *mutatis mutandis*, from Proposition 1 in Ligon, Thomas and Worrall (2002).

Proposition 3 *Given a discount factor β, for each state s there exists an interval $\left[\underline{\lambda}(\beta,s),\bar{\lambda}(\beta,s)\right]$ such that if the state realised in period $t+1$ is r, then*

(i) If $\lambda(h_t)<\underline{\lambda}(\beta,r)$, then $\lambda(h_{t+1})=\underline{\lambda}(\beta,r)$.

(ii) If $\lambda(h_t)>\bar{\lambda}(\beta,r)$, then $\lambda(h_{t+1})=\bar{\lambda}(\beta,r)$.

(iii) If $\lambda(h_t)\in\left[\underline{\lambda}(\beta,r),\bar{\lambda}(\beta,r)\right]$, then $\lambda(h_{t+1})=\lambda(h_t)$.

Proof. Define $\underline{\lambda}(\beta,r)$ implicitly using the following equation:

$$\mu^{A}\left(y_{r}^{A},y_{r}^{B},\underline{\lambda}(\beta,r)\right)+\beta w^{A}\left(\underline{\lambda}(\beta,r)\right)=d_{aut}^{A}+\beta v_{aut}^{A}$$

and $\bar{\lambda}(\beta,r)$ according to

$$\mu^{B}\left(y_{r}^{A},y_{r}^{B},\bar{\lambda}(\beta,r)\right)+\beta P\left(w^{A}\left(\bar{\lambda}(\beta,r)\right)\right)=d_{aut}^{B}+\beta v_{aut}^{B}$$

where $w^{A}(\lambda)=P'^{-1}(-\lambda)$.

(i) Consider, first, the case where $\lambda(h_t)<\underline{\lambda}(\beta,r)$. Then, according to Lemma (2), $\theta_{s}^{A}>0$. This implies

$$\mu^{A}\left(y_{r}^{A},y_{r}^{B},\lambda(h_{t+1})\right)+\beta w^{A}\left(\lambda(h_{t+1})\right)=d_{aut}^{A}+\beta v_{aut}^{A}$$

Therefore, we obtain $\lambda(h_{t+1})=\underline{\lambda}(\beta,r)$.

(ii) Similarly, if $\lambda(h_t)>\bar{\lambda}(\beta,r)$, then Lemma (2) states that $\theta_{s}^{B}>0$. Then, Person B's participation constraint must bind and we obtain

$$\mu^{B}\left(y_{r}^{A},y_{r}^{B},\lambda(h_{t+1})\right)+\beta P\left(w^{A}\left(\lambda(h_{t+1})\right)\right)=d_{aut}^{B}+\beta v_{aut}^{B}$$

Therefore, we obtain $\lambda(h_{t+1}) = \bar{\lambda}(\beta, r)$.

(iii) Directly from Lemma (2), we obtain $\lambda(h_{t+1}) = \lambda(h_t)$. ■

The proof of Lemma 6 makes use of the following lemma.

Lemma 5 *:$\underline{\lambda}(\beta, s)$ is decreasing in β; $\bar{\lambda}(\beta, s)$ is increasing in β.*

Proof. Suppose that person A's pareto weight in some period t, $\lambda(t)$, when state s is realised, equals $\underline{\lambda}(\beta, s)$. We partition the possible states of nature as follows:

$$\begin{aligned}
\mathcal{S}^A &= \{\hat{s} \in \mathcal{S} : \underline{\lambda}(\beta, \hat{s}) \geq \underline{\lambda}(\beta, s)\} \\
\mathcal{S}^0 &= \left\{\hat{s} \in \mathcal{S} : \underline{\lambda}(\beta, \hat{s}) < \underline{\lambda}(\beta, s) \leq \bar{\lambda}(\beta, \hat{s})\right\} \\
\mathcal{S}^B &= \left\{\hat{s} \in \mathcal{S} : \bar{\lambda}(\beta, \hat{s}) < \underline{\lambda}(\beta, s)\right\}
\end{aligned}$$

By definition, we have

$$\mu^A(\underline{\lambda}(\beta, s), y_s) + \beta w^A(\beta, \underline{\lambda}(\beta, s)) = d_s^A + \beta v_{aut}^A$$

$$\begin{aligned}
&\Longrightarrow \quad \mu^A(\underline{\lambda}(\beta, s), y_s) + \beta \sum_{r \in \mathcal{S}} \left[\mu^A(\lambda(r), y_r) + \beta w^A(\beta, \lambda(r))\right] = d_s^A + \beta \sum_{r \in \mathcal{S}} \left[d_r^A + \beta v_{aut}^A\right] \\
&\Longrightarrow \quad \mu^A(\underline{\lambda}(\beta, s), y_s) + \beta \sum_{r \in \mathcal{S}^0 + \mathcal{S}^B} \left[\mu^A(\lambda(r), y_r) + \beta w^A(\beta, \lambda(r))\right] = d_s^A + \beta \sum_{r \in \mathcal{S}^0 + \mathcal{S}^B} \left[d_r^A + \beta v_{aut}^A\right] \\
&\Longrightarrow \quad \mu^A(\underline{\lambda}(\beta, s), y_s) - d_s^A + \beta \sum_{r \in \mathcal{S}^0 + \mathcal{S}^B} \left[\mu^A(\lambda(r), y_r) - d_r^A + \beta\left(w^A(\beta, \lambda(r)) - v_{aut}^A\right)\right] = 0 \qquad (32)
\end{aligned}$$

For $r \in \mathcal{S}^0$, we have, by definition of $\mathcal{S}^0$ and Proposition 3,

$$\mu^A(\lambda(r), y_r) + \beta w^A(\beta, \lambda(r)) = \mu^A(\underline{\lambda}(\beta, s), y_r) + \beta w^A(\beta, \underline{\lambda}(\beta, s))$$

$$\begin{aligned}
\Longrightarrow \quad & \mu^A(\lambda(r), y_r) + \beta w^A(\beta, \lambda(r)) - d_r^A - \beta v_{aut}^A \\
= \quad & \mu^A(\underline{\lambda}(\beta, s), y_r) + \beta w^A(\beta, \underline{\lambda}(\beta, s)) - d_r^A - \beta v_{aut}^A \\
= \quad & \mu^A(\underline{\lambda}(\beta, s), y_r) - \mu^A(\underline{\lambda}(\beta, s), y_s) - d_r^A + d_s^A
\end{aligned}$$

Then, the surplus is independent of β for $r \in \mathcal{S}^0$. Finally, for $r \in \mathcal{S}^B$, the continuation value to person A equals

$$\begin{aligned}
\max \quad &: \quad u^A\left(x^A, z\right) + \beta E \sum_{t=1}^{\infty} \beta^{t-1} u^A\left(x^A, z\right) \qquad (33) \\
\text{s.t.} \quad &: \quad u^B\left(x^B, z\right) + \beta E \sum_{t=1}^{\infty} \beta^{t-1} u^B\left(x^B, z\right) \geq d_r^B + \beta v_{aut}^B
\end{aligned}$$

Note further that the solution to the maximisation problem in (33) is the same as that for the following:

$$\begin{aligned} \max \quad &: \quad u^A\left(x^A, z\right) - d^A + \beta E \sum_{t=1}^{\infty} \beta^{t-1}\left[u^A\left(x^A, z\right) - d^A\right] \\ \text{s.t.} \quad &: \quad u^B\left(x^B, z\right) + \beta E \sum_{t=1}^{\infty} \beta^{t-1} u^B\left(x^B, z\right) \geq d_r^B + \beta v_{aut}^B \end{aligned} \tag{34}$$

as we have merely added constants to the maximand for the latter. Then, the surplus must be non-decreasing in β because the constraint set is expanding in β and the maximand is increasing in β. Thus, differentiating throughout (32) w.r.t β, shows that $\underline{\lambda}(\beta, s)$ is non-increasing in β because the surplus is independent of β for $r \in \mathcal{S}^0$ and it is non-decreasing in β for $r \in \mathcal{S}^B$. Using analogous arguments, we can show that $\bar{\lambda}(\beta, s)$ is non-decreasing in β. ■

The proof of Proposition 2 makes use of the following lemma.

Lemma 6 *:If $\beta' < \beta$, then, at each λ, $\frac{\partial w^i(\beta', \lambda)}{\partial \lambda} \leq \frac{\partial w^i(\beta, \lambda)}{\partial \lambda}$.*

Proof. Using Proposition 3, we can write

$$w^i(\beta, \lambda) \equiv E \sum_{t=1}^{\infty} \beta^{t-1} \mu^i\left(y_t^A, y_t^B, \lambda_t(\beta, \lambda, h_t)\right) \tag{35}$$

where

$$\begin{aligned} \lambda_1(\beta, \lambda, s) &= \underline{\lambda}(\beta, s) \text{ if } \lambda < \underline{\lambda}(\beta, s) \\ &= \lambda \text{ if } \underline{\lambda}(\beta, s) \leq \lambda \leq \bar{\lambda}(\beta, s) \\ &= \bar{\lambda}(\beta, s) \text{ if } \lambda > \bar{\lambda}(\beta, s) \end{aligned}$$

and, for $t > 1$, we have

$$\begin{aligned} \lambda_t(\beta, \lambda, h_t) &= \underline{\lambda}(\beta, s) \text{ if } \lambda_{t-1}(\beta, \lambda, h_{t-1}) < \underline{\lambda}(\beta, s) \\ &= \lambda \text{ if } \underline{\lambda}(\beta, s) \leq \lambda_{t-1}(\beta, \lambda, h_{t-1}) \leq \bar{\lambda}(\beta, s) \\ &= \bar{\lambda}(\beta, s) \text{ if } \lambda_{t-1}(\beta, \lambda, h_{t-1}) > \bar{\lambda}(\beta, s) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial \lambda_t}{\partial \lambda}(\beta, \lambda, h_t) &= \frac{\partial \lambda_{t-1}}{\partial \lambda}(\beta, \lambda, h_{t-1}) \text{ if } \underline{\lambda}(\beta, s) \leq \lambda_{t-1}(\beta, \lambda, h_{t-1}) \leq \bar{\lambda}(\beta, s) \\ &= 0 \text{ otherwise} \end{aligned}$$

Thus, by iteration we obtain

$$\begin{aligned} \frac{\partial \lambda_t}{\partial \lambda}(\beta, \lambda, h_t) &= 1 \text{ if } \lambda \in \left[\underline{\lambda}(\beta, s), \bar{\lambda}(\beta, s)\right] \text{ for each } s \text{ contained in } h_t \\ &= 0 \text{ otherwise} \end{aligned}$$

Denote by $\rho_t(\beta,\lambda)$ the probability that $\lambda \in \left[\underline{\lambda}(\beta, s_\tau), \bar{\lambda}(\beta, s_\tau)\right]$ for $\tau = 1..t$, where s_τ is the state realised in period τ. Then, differentiating throughout (35) w.r.t. to λ, we obtain

$$\frac{\partial w^i}{\partial \lambda}(\beta,\lambda) = \sum_{t=1}^{\infty} \beta^{t-1} E \frac{\partial \mu^i}{\partial \lambda_t} \rho_t(\beta,\lambda) \tag{36}$$

By Lemma 5, $\underline{\lambda}(\beta, s)$ is decreasing in β, and $\bar{\lambda}(\beta, s)$ is increasing in β. Therefore, $\rho_t(\beta,\lambda)$ is (weakly) increasing in β. Therefore, using (36), $\frac{\partial w^i}{\partial \lambda}(\beta,\lambda)$ is (weakly) increasing in β. ■

Proof. of Proposition 2: (i) From the fundamental theorem of calculus,

$$x^B\left(y_s^A, y_s^B\right) = x^B\left(y_s^A, y_r^B\right) + \int_{y_r^B}^{y_s^B} \frac{\partial x^B\left(y^A, y^B\right)}{\partial y^B} dy^B$$

Since $y_s^B = y_r^B$, we have, using Lemma 3, $x^B\left(y_r^A, y_r^B\right) < x^B\left(y_s^A, y_s^B\right) \implies x_r^B[A] < x_s^B[A]$. Likewise, we can show that $z_r[A] < z_s[A]$.

(ii) To find sufficient conditions for $x_r^A[A] > x_s^A[A]$, we proceed as follows. Rearranging (17), we obtain

$$\beta\frac{\partial \bar{w}}{\partial \lambda} \geq \left(u_2^A(..)\left(\frac{\partial \bar{x}^A}{\partial \lambda}\frac{\partial \bar{z}}{\partial y^B} - \frac{\partial \bar{x}^A}{\partial y^B}\frac{\partial \bar{z}}{\partial \lambda}\right) - \frac{\partial \bar{x}^A}{\partial \lambda}\frac{\partial d^A}{\partial y^B}\right) / \frac{\partial \bar{x}^A}{\partial y^B} \tag{37}$$

By Lemma 4, for each $y^A, y^B, \lambda \in (0,\infty)$,

$$\frac{\partial \bar{x}^A}{\partial \lambda}\frac{\partial \bar{z}}{\partial y^B} - \frac{\partial \bar{x}^A}{\partial y^B}\frac{\partial \bar{z}}{\partial \lambda} > 0$$

Then, for $\frac{\partial d^A}{\partial y^B}$ sufficiently small, then the right-hand side of the inequality in (37) is greater than zero. Define $K\left(y^A, y^B\right)$ as follows:

$$K\left(y^A, y^B\right) = \min_{\lambda \in \{\underline{\lambda}(y^A, y^B, \beta):\beta \in (0,1)\}} \left[\left(u_2^A(..)\left(\frac{\partial \bar{x}^A}{\partial \lambda}\frac{\partial \bar{z}}{\partial y^B} - \frac{\partial \bar{x}^A}{\partial y^B}\frac{\partial \bar{z}}{\partial \lambda}\right) - \frac{\partial \bar{x}^A}{\partial \lambda}\frac{\partial d^A}{\partial y^B}\right) / \frac{\partial \bar{x}^A}{\partial y^B}\right]$$

where the expression within the square brackets is computed for $\left(y^A, y^B\right)$ and λ. Further define

$$\hat{w}^i(\beta) = \max\left\{\frac{\partial w^i(\beta)}{\partial \lambda} : \lambda \in \left[\underline{\lambda}^*(\beta), \bar{\lambda}^*(\beta)\right]\right\} \tag{38}$$

$$\beta\left(y^A, y^B\right) = \max\left\{\beta : \beta\hat{w}^i(\beta) < K\left(y^A, y^B\right)\right\} \tag{39}$$

$$\beta^* = \min_{y^B \in [y_s^B, y_r^B]} \beta\left(y^A, y^B\right) \tag{40}$$

where $\underline{\lambda}^*(\beta) = \min\{\underline{\lambda}(\beta, s)\}_{s\in\mathcal{S}}$ and $\bar{\lambda}^*(\beta) = \max\left\{\bar{\lambda}(\beta, s)\right\}_{s\in\mathcal{S}}$. According to Lemma 6, $\hat{w}^i(\beta)$ is (weakly) increasing in β. Therefore, the set on the right-hand side of (39) is non-empty. Then $\beta^* > 0$. Furthermore, by construction, the inequality in (37) is satisfied for each $y^B \in \left[y_s^B, y_r^B\right]$ for any $\beta < \beta^*$. Thus, by Lemma 3 and the fundamental theorem of calculus, $x^A\left(y_r^A, y_r^B\right) > x^A\left(y_s^A, y_s^B\right) \implies x_r^A[A] > x_s^A[A]$ for $\beta < \beta^*$. ■

Proposition 4 *If the participation constraint is binding for A in state s, period t and the realised state in period t + 1, s^* is such that $y_{s^*}^A = y_s^A$, $y_{s^*}^B < y_s^B$, then the participation constraint is binding for A again in period t + 1.*

Proof. of Proposition 4: Suppose the participation constraint for A does not bind in period $t+1$. Then his promised value in period $t+1$ is no larger than that in period t: $w(t+1) \leq w(t)$. And, by Assumption 2,

$$u^A\left(x^A(t+1), z(t+1)\right) - d^A\left(y^A_{s^*}, y^B_{s^*}\right) < u^A\left(x^A(t), z(t)\right) - d^A\left(y^A_s, y^B_s\right) \tag{41}$$

As A's participation constraint binds in period t, we have

$$u^A\left(x^A(t), z(t)\right) - d^A\left(y^A_s, y^B_s\right) = \beta\left(v^A_{aut} - w(t)\right)$$

Then equation (41) and $w(t+1) \leq w(t)$ implies

$$u^A\left(x^A(t+1), z(t+1)\right) - d^A\left(y^A_{s^*}, y^B_{s^*}\right) < \beta\left(v^A_{aut} - w(t+1)\right)$$

which violates A's participation constraint in period $t+1$. Therefore, the participation constraint must bind for A in period $t+1$. ■

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